3 C*-Algebras

In the same sense as Banach algebras may be seen as an abstraction of the space of bounded operators on a Banach space, we can abstract the concept of bounded operators on a Hilbert space. Of course, a Hilbert space is in particular a Banach space. So the algebras we are looking for are in particular Banach algebras. The additional structure of interest coming from Hilbert spaces is that of an *adjoint*.

Definition 3.1. Let A be an algebra over $\mathbb{C}$. Consider a map $^* : A \to A$ with the following properties:

- $(a+b)^* = a^* + b^*$ for all $a, b \in A$.
- $(\lambda a)^* = \overline{\lambda} a^*$ for all $\lambda \in \mathbb{C}$ and $a \in A$.
- $(ab)^* = b^* a^*$ for all $a, b \in A$.
- $(a^*)^* = a$ for all $a \in A$.

Then, * is called an *(antilinear antimultiplicative) involution*.

Definition 3.2. Let A be a Banach algebra with involution $^* : A \to A$ such that $\|a^*a\| = \|a\|^2$. Then, A is called a *C*-algebra*. For an element $a \in A$, the element a^* is called its *adjoint*. If $a^* = a$, then a is called *self-adjoint*. If $a^*a = aa^*$, then a is called *normal*.

Exercise 1. Let A be a C*-algebra. (a) Show that $\|a^*\| = \|a\|$ and $\|aa^*\| = \|a\|^2$ for all $a \in A$. (b) If $e \in A$ is a unit, show that $e^* = e$ and that $\|e\| = 1$. (c) If $a \in A$ is invertible, show that a^* is also invertible.

Exercise 2. Let A be a unital C*-algebra and $a \in A$. Show that $\sigma_A(a^*) = \overline{\sigma_A(a)}$.

Exercise 3. Let X be a Hilbert space. (a) Show that $BL(X,X)$ is a unital C*-algebra. (b) Show that $CP(X,X)$ is a C*-ideal in $BL(X,X)$.

Exercise 4. Let A be a C*-algebra and $a \in A$. Show that there is a unique way to write $a = b + \mathrm{i}c$ so that a and b are self-adjoint.

Exercise 5. Let T be a compact topological space. Show that the Banach algebra $C(T, \mathbb{C})$ of Exercise 3 in Section 1 is a C*-algebra, where the involution is given by complex conjugation.

Proposition 3.3. *Let A be a C*-algebra and $a \in A$ normal. Then,* $\|a^2\| = \|a\|^2$ *and* $r_A(a) = \|a\|$.

Proof. We have $\|a^2\|^2 = \|(a^2)^*(a^2)\| = \|(a^*a)^*(a^*a)\| = \|a^*a\|^2 = (\|a\|^2)^2$. This implies the first statement. Also, this implies $\|a^{2^k}\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$ and hence $\lim_{n\to\infty} \|a^n\|^{1/n} = \|a\|$ if the limit exists. But by Proposition 1.12 the limit exists and is equal to $r_A(a)$. $\square$

Proposition 3.4. *Let A be a C^*-algebra and $a \in A$ self-adjoint. Then, $\sigma_A(a) \subset \mathbb{R}$.*

Proof. Take $\alpha + \mathrm{i}\beta \in \sigma_A(a)$, where $\alpha, \beta \in \mathbb{R}$. Thus, for any $\lambda \in \mathbb{R}$ we have $\alpha + \mathrm{i}(\beta + \lambda) \in \sigma_A(a + \mathrm{i}\lambda e)$. By Proposition 1.7 we have $|\alpha + \mathrm{i}(\beta + \lambda)| \leq \|a + \mathrm{i}\lambda e\|$. We deduce

$$\begin{aligned} \alpha^2 + (\beta + \lambda)^2 &= |\alpha + \mathrm{i}(\beta + \lambda)|^2 \\ &\leq \|a + \mathrm{i}\lambda e\|^2 \\ &= \|(a + \mathrm{i}\lambda e)^*(a + \mathrm{i}\lambda e)\| \\ &= \|(a - \mathrm{i}\lambda e)(a + \mathrm{i}\lambda e)\| \\ &= \|a^2 + \lambda^2 e\| \\ &\leq \|a^2\| + \lambda^2 \end{aligned}$$

Substracting λ^2 on both sides we are left with $\alpha^2 + \beta^2 + 2\beta\lambda \leq \|a^2\|$. Since this is satisfied for all $\lambda \in \mathbb{R}$ we conclude $\beta = 0$. □

Lemma 3.5 (Complex Stone-Weierstrass Theorem)**.** *Let T be a compact topological space and $A \subseteq C(T, \mathbb{C})$ a subalgebra of the algebra of complex valued continuous functions over T. Assume that A separates points, contains the constant functions and is self-conjugate. Then the closure of A in $C(T, \mathbb{C})$ is equal to $C(T, \mathbb{C})$.*

For a proof, see e.g., the book of Lang.

Theorem 3.6 (Gelfand-Naimark)**.** *Let A be a unital commutative C^*-algebra. Then, the Gelfand transform $A \to C(\Gamma_A, \mathbb{C})$ is an isomorphism of unital commutative C^*-algebras.*

Proof. Since A is commutative all its elements are normal. Then, by Proposition 3.3, $\|a^2\| = \|a\|^2$ and we can apply Proposition 2.17. So, we know that the Gelfand transform is an isomorphism of unital commutative Banach algebras onto its image $\hat{A} \subseteq C(\Gamma_A, \mathbb{C})$.

We proceed to show that the Gelfand transform respects the involution. Let $a \in A$ be self-adjoint. Then, combining Proposition 2.15 with Proposition 3.4 we get $\hat{a}(\phi) = \phi(a) \in \sigma_A(a) \subset \mathbb{R}$ for all $\phi \in \Gamma_A$. So $\hat{a}$ is real-valued, i.e., self-adjoint. In particular, $\widehat{a^*} = \hat{a}^*$. Using the decomposition of Exercise 4 this follows for general elements of A. **(Explain!)**

It remains to show that $\hat{A} = C(\Gamma_A, \mathbb{C})$. We apply Lemma 3.5. The fact that $\hat{A}$ separates points is true by construction, that $\hat{A}$ contains the constant functions follows from unitality and self-conjugacy of $\hat{A}$ is a consequence of the fact that $\hat{A}$ is the image of a C*-algebra homomorphism. Furthermore, $\hat{A}$ is closed as we have already seen. □